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ON THE OSCILLATIONS NEAR AND AT RESONANCE IN OPEN PIPES

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SUMMARY

This paper is concerned with resonance oscillations occurring when a piston executes small oscillations on one end of a pipe which is open to the atmosphere at the other end. According to linear theory very large amplitudes of pressure and velocity oscillations in the gas in the pipe result when the piston is oscillated with an angular frequency near $\pi a_0/2L$, where a_0 is the sound velocity of the gas and L the length of the pipe. In the theory of resonators, due to Helmholtz and Rayleigh and discussed in section 1, radiation from the open end is taken into account. Then resonance occurs at a frequency slightly below ω_0, and amplitudes are still very large, as is shown in section 1. Therefore a nonlinear theory is developed here, analogous to previous work on resonance oscillations in closed pipes. In section 2 the boundary conditions at the open end are formulated based on the fact that the reservoirconditions are constant at inflow but vary at outflow, since the gas issues as a jet. This difference results in a net efflux of energy to be balanced by the work done by the piston. In sections 3-7 a perturbation theory is developed in terms of the characteristics of motion. The pertinent perturbation parameter is suggested by the energy balance. An ordinary differential equation for the first order perturbation in the quasi-steady state is obtained in section 7. In section 8 experimental results are presented together with results obtained from numerical integration of the above mentioned equation. The results, showing a satisfactory agreement, indicate that further experimental investigation on the conditions at the open end are needed.

1. *Introduction*.

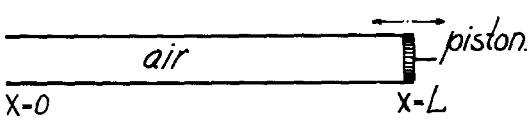


Figure 1: Open pipe with oscillating piston at one end.

A circular pipe with cross section πR^2 and length L is open to the atmosphere at $x = 0$ (see figure 1). At $x = L$ a piston executes harmonic oscillations with amplitude δ and angular frequency ω. When δ/L is very small acoustic theory may be used to find the disturbance velocity u and the disturbance pressure $p - p_0$ due to a piston displacement $\delta \cos \omega t$. The result is, (see e.g. (1)),

$$u = \frac{-\omega\delta \sin \omega t \cos \dfrac{\omega x}{a_0}}{\cos \dfrac{\omega L}{a_0}}, \tag{1.1}$$

$$\frac{p - p_0}{p_0} = \gamma \frac{\delta\omega}{a_0} \frac{\cos \omega t \sin \dfrac{\omega x}{a_0}}{\cos \dfrac{\omega L}{a_0}}. \tag{1.2}$$

a_0 is the sound velocity in the air and γ the ratio between specific heats. When ω attains the value

$$\omega_o = \frac{\pi}{2}\frac{a_o}{L}, \tag{1.3}$$

(1.1) and (1.2) yield infinite values. The reason for this is that whereas for $\omega < \omega_o$ and $\omega > \omega_o$ u and $p - p_o$ are 90^o and -90^o out of phase at the piston, they are in phase at $\omega = \omega_o$. The piston therefore does a nonzero amount of work on the gas. Since this is not balanced by any energy consuming mechanism, amplitudes continue to grow. This is contrary to observation. In the classical theory of resonators due to Helmholtz and Rayleigh, described in the latter's "The Theory of Sound"(1), this problem is solved by invoking mechanisms which are negligible in a linear theory.

The mechanism which keeps amplitudes finite at resonance is in Rayleigh's theory the radiation of energy from the open end. At some distance this may be considered as the site of a source of strenght $\pi R^2 u$. On the basis of acoustical theory the velocity potential is accordingly

$$\frac{R^2 u\,(t - r/a_o)}{4r}. \tag{1.4}$$

At values of r, such that $R \ll r \ll L$, the velocity $\frac{\partial\phi}{\partial r}$ is

$$\frac{\partial\phi}{\partial r} = -\frac{R^2\dot{u}\,(t - r/a_o)}{r\,a_o}.$$

The mean value of the radiated energy E is

$$\rho\, a_o \int\left(\frac{\partial\phi}{\partial r}\right)^2 ds,$$

where the integration is over a sphere of radius r.
Hence

$$E \sim \frac{\rho\, R^4\, \omega^2 \bar{u}^2}{a_o},\ \bar{u} \text{ being the amplitude of u.}$$

The mean work done by the piston is with a piston velocity $\omega\delta$ and a disturbance pressure $p_o u/a_o$, of the order $p_o\omega\delta\bar{u}/a_o$.
Making use of (1.3) and

$$\left(\frac{dp}{d\rho}\right)_o = \frac{\gamma p_o}{\rho_o} = a_o^{\,2}, \tag{1.5}$$

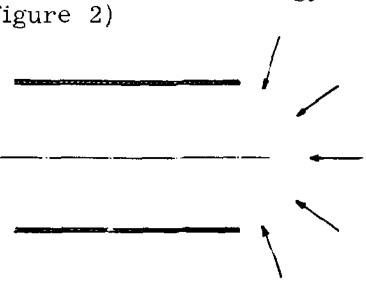
we obtain from the energy balance

$$\frac{\bar{u}}{\omega\delta} = 0\left(\frac{L}{R}\right)^2. \tag{1.6}$$

In this way finite, though still very large, amplitudes are obtained. In fact the amplitudes of velocity and pressure as predicted by this classical theory are many times larger than actually measured. With $(R/L)^2$ of order δ/L it follows from (1.6) and (1.3) that in air velocities of several hunderds m/s can be expected. In practice they are of order 1 m/s, which is still very large with respect to the velocities at off resonance frequencies, which are with $\delta = 0(10^{-3}\,m)$ and $L = 0(1m)$ typically of order of cm/sec. With this in mind the resonance phenomenon is treated in this paper as a nonlinear one, using the method of characteristics. This has been a successful approach in the case of resonance in closed pipes. It has been shown both theoretically and experimentally that in that case, and in the analogous case of oscillations of fluid with a free surface in a container, shock waves (Chu and Ying[2]) resp. hydraulic jumps (Verhagen and Van Wijngaarden[3]) are formed. The dissipation in these discontinuities is balanced by the work done by the piston and this balance determines the amplitude of the oscillations. Before proceeding we have to conceive an idea how such a balance is obtained in the case of open pipes.

2. *The energy balande for open pipes at resonance; conditions at open end.*

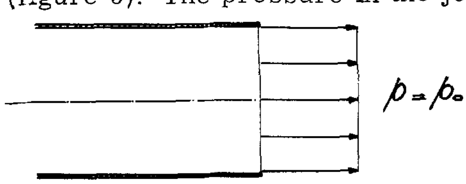
The occurrence of shock waves of appreciable strength is excluded by the presence of the open end. They are not observed in experiments. So here ceases the analogy with the closed pipe. Consider the exit of the tube. (figure 2)



When air is sucked in the pipe the motion in the surrounding air is as if a sink was located at the mouth of the pipe.

Figure 2: Motion at open end, when air is sucked in the pipe.

In the classical theory the same type of flow, with reversed sign of arrows, is assumed when air is flowing out. The velocity potential is found by matching the velocity from a potential of the type (1.4) with the velocity in the pipe. With an inviscid fluid this is a reasonable picture of the flow. A real fluid however issues from the pipe as a jet of nearly circular form (figure 3). The pressure in the jet is very nearly equal to ambient pressure.



The difference between out- and inflow is that during inflow the reservoirpressure is constant, whereas this varies during outflow, the mean value being of order $\rho_0 \bar{u}^2$ above ambient pressure. This means that a net amount of energy of order $\rho_0 \bar{u}^3$ leaves the pipe per unit time. Equating to the amount of work delivered by the piston yields

Figure 3: Motion at open end when air is issuing.

$$\rho_o \bar{u}^3 \sim \frac{p_o \bar{u}}{a_o} \omega \delta, \tag{2.1}$$

or

$$\bar{u} = 0(\frac{\delta}{L})^{\frac{1}{2}} (a_o \omega_o L)^{\frac{1}{2}}. \tag{2.2}$$

With $(R/L)^2 = 0(\delta/L)$ this is (cf 1.6) smaller by a factor of order $(\delta/L)^{\frac{1}{2}}$ than the velocity following from the balance between radiated energy and work done by the piston.

Next we formulate the boundary conditions at x = 0 on the basis of the above given flow picture.

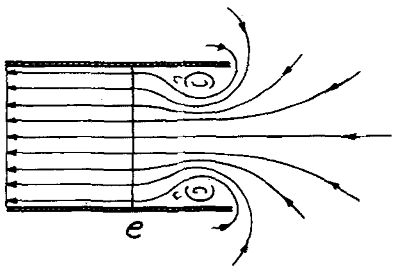


Figure 4: Sketch of flow at inlet of the pipe when air is coming in.

First we discuss the conditions at inflow, $u_e > 0$. The flow will be as sketched in figure 4.

Due to the sharp edges there is separation of the flow with associated eddy formation. After contracting the flow occupies the whole cross section at station e. For our purpose we assume that this coincides with x = 0, that is to say, we shall neglect the difference between x = 0 and the actual location of e.

To obtain a relation at x = 0 between u_e and p_e a momentum consideration is most convenient. Consider a control surface as in figure 5.

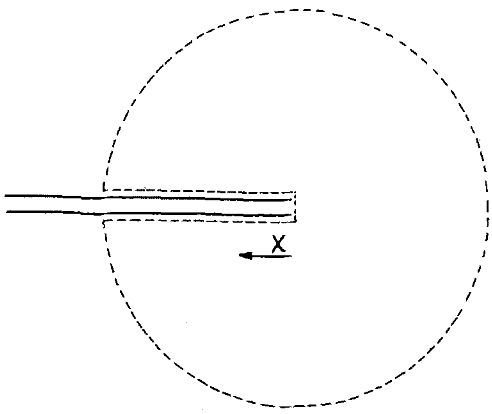


Figure 5: Controle volume for which momentum conservation is formulated.

Conservation of momentum in x direction yields

$$(p_e + \rho_e u_e^2)\,\pi R^2 + \frac{\partial}{\partial t}\int \rho \boldsymbol{u}.\boldsymbol{i}\, dV = p_o, \tag{2.3}$$

where i is a unit vector in x-direction. The integral can by making use of a velocity potential, defined by $\boldsymbol{u} = \Delta\phi$ be written as

$$\frac{\partial}{\partial t}\int \rho \boldsymbol{u}\, \boldsymbol{i}\, dV = \frac{\partial}{\partial t}\int \rho\phi\, \boldsymbol{ds}.\boldsymbol{i} - \frac{\partial}{\partial t}\int \phi\, \frac{\partial \rho}{\partial x}\, dV. \tag{2.4}$$

The first integral can be interpreted as the external mass of air to be accelerated when the piston oscillates. The second integral is associated with radiated wave momentum. For our present purposes we neglect the radiation, being of negligible order of magnitude. The motion within the control volume is as if a sink of strength $\pi R^2 u_e$ was located at the centre of the mouth, so ϕ may be written as

$$\phi \sim \frac{R^2 u_e}{4r}. \tag{2.5}$$

Due to symmetry, the surface of the large sphere does not contribute to the surface integral in (2.4). The only contribution stems from the cross section of the pipe exit and is

$$\frac{c\pi R^3}{2}\frac{\partial}{\partial t}(\rho_e u_e),$$

where c is a constant of unit order.
Introducing this in (2.5) yields

$$p_e + \rho_e u_e^2 + \frac{cR}{2}\cdot\frac{\partial}{\partial t}(\rho_e u_e) = p_o, \quad u_e > 0. \tag{2.6}$$

The precise determination of c would require the solution of a rather difficult boundary value problem. We can however avoid this by recognizing that eventually the instationary term in (2.6) determines the "virtual" or extra mass of air to be accelerated by the piston. Its effect is as if the pipe were connected with an environment of zero mass but were larger with an amount ΔL. The value of ΔL for resonators is discussed at length in ref. 1 and we shall in numerical calculations assume such a value for c that the extra length ΔL is in agreement with measurements reported in ref. 1.

Whereas (2.6) holds when air is flowing into the pipe, we have at outflow simply

$$p_e = p_o, \quad u_e < 0. \tag{2.7}$$

We conclude this section by noting that Rayleigh himself has envisaged the jet formation at outflow. In section 322 of reference 1 he writes: "It is clear that, if the formation of jets took place to any considerable extent during the passage of air through the mouths of resonators our calculations of pitch would have to be seriously modified". Later in the same section he concludes by assuming jet formation as unlikely because "the near agreement between the observed and the calculated pitch is almost a sufficient proof of this". We shall see that the occurrence of jets at the outflow does not affect the *pitch* of the resonator, but only the *amplitude* of the oscillations, a quantity difficult to measure in Rayleigh's time.

3. *Calculation of the nonlinear oscillations.*

The configuration being as in figure 1, the motion is with the symbols used in the preceding sections described by

$$\left\{\frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x}\right\}\left\{u \pm \frac{2a}{\gamma-1}\right\} = 0, \tag{3.1}$$

since due to the neglect of the effects of viscosity and heat conduction the gas motion is isentropic.

Because a perturbation method in the physical, or x-t, plane starting with the linear approximation in which $u = 0$ and $a = a_0$ leads to singular behaviour at resonance we attempt following earlier work (reference 4) a perturbation in the characteristic plane, the characteristics β and α being given by (from 3.1)

$$\frac{\partial x}{\partial \alpha} = (u + a)\frac{\partial t}{\partial \alpha} \tag{3.2}$$

$$\frac{\partial x}{\partial \beta} = (u - a)\frac{\partial t}{\partial \beta}. \tag{3.3}$$

Along the β characteristics given by (3.2)

$$\frac{\partial u}{\partial \alpha} = -\frac{2}{\gamma-1}\frac{\partial a}{\partial \alpha}, \tag{3.4}$$

whereas along the α characteristics given by (3.3)

$$\frac{\partial u}{\partial \beta} = \frac{2}{\gamma-1}\frac{\partial a}{\partial \beta}. \tag{3.5}$$

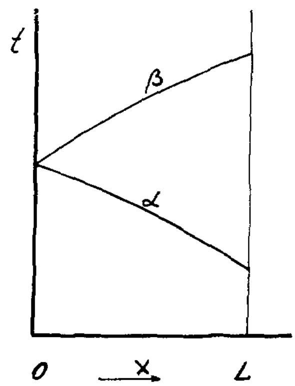


Figure 6: x, t plane with characteristics α and β.

We label the α characteristics with the value of t at the intersection with the line x = 0 (see figure 6), the β characteristics with the value of α at this intersection. Hence

$$\alpha = \beta \text{ at } x = 0 \tag{3.6}$$

$$\alpha = t \text{ at } x = 0 \tag{3.7}$$

We expand u, a, x and t in terms* of the small quantity

$$\epsilon = \left(\frac{\delta}{L}\right)^{\frac{1}{2}}, \tag{3.8}$$

which is suggested by the energy considerations in section 2. (cf. 2.2) So we have

$$u = \epsilon u_1(\alpha,\beta) + \epsilon^2 u_2(\alpha,\beta) + \ldots\ldots, \tag{3.9}$$

$$a = a_o + \epsilon a_1(\alpha,\beta) + \epsilon^2 a_2(\alpha,\beta) + \ldots\ldots, \tag{3.10}$$

$$x = x_o + \epsilon x_1(\alpha,\beta) + \epsilon^2 x_2(\alpha,\beta) + \ldots\ldots, \tag{3.11}$$

$$t = t_o + \epsilon t_1(\alpha,\beta) + \epsilon^2 t_2(\alpha,\beta) + \ldots\ldots \tag{3.12}$$

The boundary conditions in the quasi-steady state are in terms of x and t that

$$u = -\epsilon^2 \omega L \sin \omega t \text{ at } x = L + \epsilon^2 L \cos \omega t \tag{3.13}$$

and the conditions (2.6) and (2.7) at x = 0.
Introducing

$$\frac{cR}{2L} = \sigma\epsilon, \tag{3.14}$$

(2.6) can be written as

$$\left(\frac{a}{a_o}\right)^{-\frac{2\gamma}{\gamma-1}} - 1 = \frac{\rho u^2}{p} + \frac{\sigma\epsilon L}{p}\frac{\partial}{\partial t}(\rho u),$$

where use has been made of the isentropy to express p/p_o in terms of

* For the mathematical basis (convergence etc.) see Lin[4].

a/a_0. The right hand side of this equation is obviously of order ϵ^2. Expanding the left hand side with help of (3.10) shows that

$$a_1 = 0 \text{ at } \alpha = \beta, \text{ if } u > 0. \qquad (3.15)^a$$

Using this result we find upon collecting terms of order ϵ^2:

$$\frac{2a_2}{\gamma-1} = -\frac{u_1^2}{a_0} - \frac{\sigma L}{a_0}\frac{\partial u_1}{\partial t} \text{ at } \alpha = \beta, \text{ if } u > 0. \qquad (3.15)^b$$

Expressed in terms of a_1 and a_2 the condition (2.7) is

$$a_1 = a_2 = 0 \text{ at } \alpha = \beta, \text{ if } u < 0. \qquad (3.15)^c$$

The procedure now is to insert (3.7) - (3.12) in the equations (3.2) - (3.5) and the boundary conditions (3.6), (3.7), (3.13) and (3.15).

4. *Zeroth approximation.*

Collecting like orders of ϵ gives in the lowest approximation where only terms of order ϵ^0 are retained:

$$\frac{\partial x_0}{\partial \alpha} = a_0 \frac{\partial t_0}{\partial \alpha}; \quad \frac{\partial x_0}{\partial \beta} = -a_0 \frac{\partial t_0}{\partial \beta},$$

with the boundary conditions: $x_0 = 0$, $t_0 = \alpha$ at $\alpha = \beta$.
Integration yields

$$x_0 = \frac{a_0}{2}(\alpha-\beta), \qquad (4.1)$$

$$t_0 = \tfrac{1}{2}(\alpha+\beta). \qquad (4.2)$$

5. *First approximation.*

In the next approximation terms of order ϵ^1 are retained. The equations are

$$\frac{\partial x_1}{\partial \alpha} = a_0 \frac{\partial t_1}{\partial \alpha} + (u_1 + a_1)\frac{\partial t_0}{\partial \alpha}, \qquad (5.1)$$

$$\frac{\partial x_1}{\partial \beta} = -a_0 \frac{\partial t_1}{\partial \beta} + (u_1 - a_1)\frac{\partial t_0}{\partial \beta}, \qquad (5.2)$$

$$\frac{\partial u_1}{\partial \alpha} = -\frac{2}{\gamma-1}\frac{\partial a_1}{\partial \alpha}, \qquad (5.3)$$

$$\frac{\partial u_1}{\partial \beta} = \frac{2}{\gamma-1}\frac{\partial a_1}{\partial \beta}. \qquad (5.4)$$

Both for $(3.15)^a$ *and* $(3.15)^c$ $a_1 = 0$ at $\alpha = \beta$. Further it follows from (3.6) and (3.7) that $t_1 = 0$ at $\alpha = \beta$.
Hence

$$\left.\begin{array}{l} x_1 = 0 \\ t_1 = 0 \\ a_1 = 0 \end{array}\right\} \text{ at } \alpha = \beta \qquad \begin{array}{r} (5.5) \\ (5.6) \\ (5.7) \end{array}$$

Integrating (5.1) - (5.4) subjected to these conditions yields, A being a still undetermined function:

$$u_1 = \tfrac{1}{2}\left\{A(\alpha) + A(\beta)\right\}, \tag{5.8}$$

$$a_1 = \frac{\gamma-1}{4}\left\{A(\beta) - A(\alpha)\right\}, \tag{5.9}$$

$$x_1 = \frac{1}{16}\left[(\gamma+1)(\alpha-\beta)\left\{A(\beta) - A(\alpha)\right\}\right], \tag{5.10}$$

$$t_1 = \frac{1}{8a_o}\left[(3-\gamma)\int_\alpha^\beta A(\xi)d\xi + (\beta-\alpha)\left\{A(\beta) - A(\alpha)\right\}\right]. \tag{5.11}$$

The conditions (3.13) give us an important property of A. In the zeroth approximation the piston is according to (3.13) and (4.1) located at $\beta = \alpha - 2L/a_o$. Here u_1 must, on account of (3.13), be zero, whence from (5.8)

$$A(\alpha) + A(\alpha - \frac{2L}{a_o}) = 0. \tag{5.12}$$

This relation has a central rôle in the theory, expressing that A is a symmetrical periodic function of α with period $4L/a_o$. In the first approximation the piston position has to be corrected with $\nu(\alpha)$, say, so that the piston is at $\beta = \alpha - 2L/a_o + \epsilon\nu(\alpha)$. The contribution of order ϵ to x there is on account of (3.11) and (5.10)

$$\nu(\alpha)\frac{\partial x_o}{\partial\beta} + \frac{(\gamma+1)L}{8a_o}\left\{A(\alpha-\frac{2L}{a_o}) - A(\alpha)\right\}.$$

From (3.13) it follows that this is zero. We obtain, using (4.1) and (5.12) $\nu(\alpha) = -\frac{(\gamma+1)L}{2a_o^2}A(\alpha)$, so that in the first approximation the piston is at

$$\beta = \alpha - \frac{2L}{a_o} - \frac{\epsilon(\gamma+1)L}{2a_o^2}A(\alpha). \tag{5.13}$$

The interesting conclusion from this section is that a continuous oscillating function $A(\alpha)$ exists, and to fix ideas we can on account of (5.8) identify $A(\alpha)$ with the first order velocity at the mouth of the pipe, which obeys all boundary conditions.

Excepted for (5.12) A(a) is undeterminate in the first order approximation. We proceed therefore to the second order.

6. *Second approximation.*

We need only the equations for the Riemann invariants which are

$$\frac{\partial u_2}{\partial\alpha} = -\frac{2}{\gamma-1}\frac{\partial a_2}{\partial\alpha}, \tag{6.1}$$

$$\frac{\partial u_2}{\partial \beta} = \frac{2}{\gamma-1}\frac{\partial a_2}{\partial \beta}. \tag{6.2}$$

Integration gives, introducing the functions B and B*,

$$u_2 = \tfrac{1}{2}\left\{B(\beta) + B^*(\alpha)\right\}, \tag{6.3}$$

$$a_2 = \frac{\gamma-1}{4}\left\{B(\beta) - B^*(\alpha)\right\}. \tag{6.4}$$

The boundary condition in (3.13) involves sin ωt. At this moment we have to express that we are concerned with frequencies near resonance and introduce, ω_0 being given by (1.3),

$$\omega = \omega_0 + \epsilon\omega_1 = \frac{\pi}{2}\frac{a_0}{L} + \epsilon\omega_1. \tag{6.5}$$

The piston position is given by (5.13). The second order term of the velocity there is

$$u_2(\alpha, \alpha - \frac{2L}{a_0}) - \frac{(\gamma+1)L}{2a_0^2} A(\alpha)\left(\frac{\partial u_1}{\partial \beta}\right)_{\alpha - \frac{2L}{a_0}}.$$

This expression must, from (3.13), equal $-\omega L \sin \omega t$, which is in this approximation, by virtue of (4.2), (6.5) and (5.12), the same as $\omega_0 L \cos \omega_0\alpha$. Equating both expressions results with help of (5.8), (5.12), and (5.13) in

$$\tfrac{1}{2}\left\{B(\alpha - \frac{2L}{a_0}) + B^*(\alpha)\right\} + \frac{(\gamma+1)L}{2a_0^2} A\frac{dA}{d\alpha} = \omega_0 L \cos \omega_0\alpha. \tag{6.6}$$

Next the conditions (3.15) must be used. The change in sign of u is to order ϵ^2 determined by the change in sign of u_1. We shall therefore choose $(3.15)^b$ when $u_1(\alpha, \alpha)$ is positive and $(3.15)^c$ when $u_1(\alpha, \alpha)$ is negative. In the latter case $a_2 = 0$ and from (6.4) it follows that, since at $\alpha = \beta$ $u_1(\alpha, \alpha) = A(\alpha)$ (cf 5.8),

$$B^* = B \text{ for } \alpha \text{ such that } A(\alpha) < 0. \tag{6.7}$$

For $A(\alpha) > 0$ $(3.15)^b$ yields with help of (5.8)

$$\frac{4a_2}{\gamma-1} = -\frac{2\sigma L}{a_0}\frac{dA}{d\alpha} - \frac{2A^2}{a_0},$$

where we have used that at $\alpha = \beta$, $\frac{\partial}{\partial t} = \frac{d}{d\alpha}$.

In combination with (6.3) we obtain that

$$B^* = B + \frac{2\sigma L}{a_0}\frac{dA}{d\alpha} + \frac{2A^2}{a_0}, \text{ for } \alpha \text{ such that } A(\alpha) > 0 \tag{6.8}$$

7. *The equation for $A(\alpha)$.*

In this section we deal with the crucial question how to obtain an equation for the first order solution $A(\alpha)$. It is worth while to note that the transition from one boundary condition at x = 0 to another introduces discontinuities in the *second order* function B. The *first* order solution $A(\alpha)$ however is continuous.

Consider some $\alpha = \alpha_1$ such that $A(\alpha_1) > 0$. For $\alpha = \alpha_1$ (6.8) holds and substitution in (6.6) gives

$$\tfrac{1}{2}\left\{B(\alpha_1 - \frac{2L}{a_o}) + B(\alpha_1)\right\} + \frac{\sigma L}{a_o}\left(\frac{dA}{d\alpha}\right)_{\alpha=\alpha_1} + \frac{A(\alpha_1)^2}{a_o} +$$

$$+ \frac{(\gamma+1)L}{2a_o^2} A(\alpha_1)\left(\frac{dA}{d\alpha}\right)_{\alpha=\alpha_1} = \omega_o L \cos \omega_o \alpha_1. \qquad (7.1)$$

Next we take $\alpha = \alpha_1 - 2L/a_o$. By virtue of (5.12) for this value of α, A is negative and (6.7) applies. Substitution of (6.7) in (6.6) gives with help of (5.12)

$$\tfrac{1}{2}\left[B(\alpha_1 - \frac{4L}{a_o}) + B(\alpha_1 - \frac{2L}{a_o})\right] + \frac{(\gamma+1)L}{2a_o^2} A(\alpha_1)\left(\frac{dA}{d\alpha}\right)_{\alpha=\alpha_1} = -\omega_o L \cos \omega_o \alpha_1.$$

Subtracting this result from (7.1) gives

$$\tfrac{1}{2}\left\{B(\alpha_1) - B(\alpha_1 - \frac{4L}{a_o})\right\} + \frac{\sigma L}{a_o}\left(\frac{dA}{d\alpha}\right)_{\alpha=\alpha_1} + \frac{A(\alpha_1)^2}{a_o} = 2\omega_o L \cos \omega_o \alpha_1.$$

We could have started also with assuming $A(\alpha_1) < 0$. Then we arrive, as may easily be verified, at

$$\tfrac{1}{2}\left\{B(\alpha_1) - B(\alpha_1 - \frac{4L}{a_o})\right\} + \frac{\sigma L}{a_o}\left(\frac{dA}{d\alpha}\right)_{\alpha=\alpha_1} - \frac{A(\alpha_1)^2}{a_o} = 2\omega_o L \cos \omega_o \alpha_1.$$

Both relations can be turned in the equation

$$\tfrac{1}{2}\left\{B(\alpha) - B(\alpha - \frac{4L}{a_o})\right\} + \frac{\sigma L}{a_o}\frac{dA}{d\alpha} + \frac{A|A|}{a_o} = 2\,\omega_o L \cos \omega_o \alpha. \qquad (7.3)$$

Finally we establish a relation between $B(\alpha)$ and $B(\alpha - 4L/a_o)$. Because all quantities have to be periodic with period $2\pi/\omega$, we have at $\beta = \alpha$

$$u(\alpha) = u\left(\alpha + \frac{2\pi}{\omega}\right). \qquad (7.4)$$

Further at $\beta = \alpha$

$$u = \epsilon A + \epsilon^2 u_2, \text{ and } \frac{2\pi}{\omega_o} = \frac{2\pi}{\omega} - \frac{2\pi\epsilon\omega_1}{\omega_o^2} \text{ so that by virtue of (5.12)} \qquad (7.5)$$

$$u\left(\alpha - \frac{2\pi}{\omega}\right) = \epsilon A(\alpha) + \epsilon^2\left\{\frac{2\pi\omega_1}{\omega_o^2}\frac{dA}{d\alpha} + u_2\left(\alpha - \frac{4L}{a_o}\right)\right\}. \qquad (7.6)$$

Using (6.3), (6.7), (6.8) and the periodic properties of $A(\alpha)$ expressed by (5.12) we obtain from (7.4) - (7.6)

$$B(\alpha) - B\left(\alpha - \frac{4L}{a_o}\right) = \frac{2\pi\omega_1}{\omega_o^2}\frac{dA}{d\alpha}. \qquad (7.7)$$

Substituting of (7. 7) into (7. 3) gives the principal result of this investigation

$$\left(\frac{\sigma L}{a_o} + \frac{\pi \omega_1}{\omega_o^2}\right) \frac{dA}{d\alpha} + \frac{A|A|}{a_o} = 2\omega_o L \cos \omega_o \alpha. \tag{7.8}$$

Introducing

$$\tau = \omega_o \alpha \tag{7.9}$$

$$A = (\omega_o L a_o)^{\frac{1}{2}} A^*, \tag{7.10}$$

this may, with help of (1. 3) be written as

$$\left(\frac{\pi}{8}\right)^{\frac{1}{2}} \left(\sigma + \frac{2\omega_1}{\omega_o^2}\right) \frac{dA^*}{d\tau} + \tfrac{1}{2} A^* \left|A^*\right| = \cos \tau. \tag{7.11}$$

A maximum of the amplitude of A^* occurs for $\frac{\omega_1}{\omega_o} = -\frac{\sigma}{2}$. According to ref. 1 an estimate for the shift in frequency with respect to ω_o is given by

$$\frac{\Delta L}{L} = 0.6 \frac{R}{L}.$$

By virtue of (1. 3) this corresponds with $\frac{\omega_o - \omega}{\omega_o} = 0.6 \frac{R}{L}$.

Then, remembering that $\omega - \omega_o = + \epsilon\omega_1 + \ldots$, we have to obtain agreement to put

$$\sigma = 1.2 \frac{R}{L} \epsilon^{-1}. \tag{7.12}$$

(This, incidentally, yields the value 2,4 for c in (3. 14)).

Equation (7. 8) can not be integrated analytically, but is in the form (7. 11) suited for integration on an analog computer. Some conclusions can be directly drawn from (7. 8). To fix ideas we note that at the piston where $\beta = \alpha - 2L/a_o + 0(\epsilon)$ we have by virtue of (5. 9) and (5. 12)

$$a_1 = -\frac{\gamma - 1}{2} A(\alpha).$$

Further, from the isentropic relation (1. 5),

$$\frac{p - p_o}{p_o} = \frac{2\gamma}{\gamma - 1} \epsilon \frac{a_1}{a_o} + 0(\epsilon^2), \text{ so that}$$

at the piston

$$\frac{p - p_o}{p_o} = -\frac{\gamma \epsilon A(\alpha)}{a_o} + 0(\epsilon^2).$$

From (4. 1), (4. 2), (5. 10) and (5. 11) it follows that at the piston

$$\alpha = t + \frac{L}{a_o} + 0(\epsilon), \tag{7.13}$$

whence up till $O(\epsilon^2)$

$$\frac{p - p_o}{p_o} = - \frac{\gamma \epsilon A(t + \frac{L}{a_o})}{a_o}. \tag{7.14}$$

When $\left|\frac{\omega_1}{\omega_o}\right|$ is sufficiently large the first term on the left hand side of (7.8) dominates and, making use of (1.3), we obtain

$$A = - \frac{\sin \omega_o \alpha}{\frac{\sigma}{2} + \frac{\omega_1}{\omega_o}}.$$

Substitution in (7.14) gives with $\alpha = t + \frac{L}{a_o} = t + \frac{\pi}{2\omega_o}$

$$\frac{p - p_o}{p_o} = - \frac{\gamma \epsilon}{\frac{\omega_1}{\omega_o} + \frac{\sigma}{2}} \cos \omega_o t. \tag{7.15}$$

Remembering that the piston displacement is $\delta \cos \omega t$ it follows that for frequencies well below (in terms of ϵ) ω_o the pressure is at the piston inphase with the displacement and at frequencies well above ω_o in antiphase. The solution (7.15) has to match with the linear solution (1.2), Indeed, evaluation of (1.2) in the neighbourhood of $\frac{\omega L}{a_o} = \frac{\pi}{2}$ gives

$$\frac{p - p_o}{p_o} \sim - \frac{\gamma \epsilon}{\frac{\omega_1}{\omega_o}} \cos \omega_o t,$$

which indicates that the linear solution and nonlinear solution become identical far enough from resonance.

Whereas linear theory predicts an infinite amplitude at $\omega = \omega_o$, the result of the present theory is that resonance, that is to say maximum of the amplitude of oscillation, occurs at a frequency given by

$$\frac{\omega_1}{\omega_o} = - \frac{\sigma}{2}, \tag{7.16}$$

with a pressure disturbance

$$\frac{p - p_o}{p_o} = \gamma \epsilon \pi^{\frac{1}{2}} \left|\sin \omega_o t\right|^{\frac{1}{2}} \operatorname{sgn} (\sin \omega_o t). \tag{7.17}$$

This result follows from (7.8) by insertion of (7.10) and by making use of (1.3) and (7.14).

At resonance there is a phase difference of $\pi/2$ between pressure at the piston and piston displacement. For those values of t for which $\sin \omega_o t = 0$, $\frac{dp}{dt}$ tends to infinite values.

This is an indication that the present theory has to be improved. Near these values of t A is very small and higher order effects may be of importance. We have neglected the difference between $x = x_e$ and $x = 0$ (see figure 4) in formulating the boundary condition at $x = 0$. Also we have applied these conditions, for $u_e > 0$ and $u_e < o$, to the approximation $A > 0$ and $A < 0$. Before it becomes worth while to carry out refinements of this type, it is necessary to know to what extent the assumed boundary conditions really apply. Mixing of incoming air with air just issued from the pipe, precise position of boundary layer separation at the exit, may significantly affect the boundary condition at the exit. Before a more accurate analysis can be carried out, experimental investigation of the conditions at the exit is needed.

Finally a difficulty must be noted occurring with the solution of (7.11) on the analog computer. We are interested in periodic solutions because the whole analysis pertains to a quasi-steady state after a long time has elapsed. Equation (7.11) does not represent the transient motion in the pipe occurring when the piston is started from rest.

These transients are of course described by the equation (3.1) but not by (7.11) since this equation holds only for the quasi-steady state. In carrying out the integration of (7.11) with an analog computer solutions of the homogeneous equation will play a rôle because one has to start from a certain initial value. These solutions have however nothing to do with transients of the physical phenomena in the pipe. How to get rid of these?
Writing (7.11) as

$$g_1 \frac{dA}{d\tau} + g_2 A|A| = \cos \tau, \tag{7.18}$$

it follows that the solution of the homogeneous equation can implicitly be given by

$$A \sim \exp\left\{ - \frac{g_2}{g_1} \int |A| \, d\tau \right\}. \tag{7.19}$$

Because $g_2 > 0$, there is no difficulty for $g_1 > 0$. However for $g_1 < 0$ (or $\frac{\omega_1}{\omega_0} + \frac{\sigma}{2} < 0$) (7.19) increases exponentially and the solution obtained with the analog computer is unstable. The difficulty is in the present case avoided by making use of the fact that in a quasi-steady state changing the sign of t means only changing the phase. Since it follows from (7.18) that changing the sign of t and that of g_1 has the same effect, a solution for $g_1 < 0$ can be obtained by first seeking the solution for $|g_1|$. Let this solution result in a pressure at the piston with phase $\pi/2 + \psi$ with respect to the piston displacement. Then the solution for g_1 has the same amplitude but phase $\pi/2 - \psi$ with respect to piston displacement. In this way the quasi-steady solution for $g_1 < 0$ was obtained.

8. *Comparison with experiments*.

Some experiments were carried out to verify the theory. From these we have selected for presentation here measurements with a pipe of L = 4.82 m, 2R = 237 x 10^{-3} m. During the measurements the atmospheric pressure p_0 was very nearly $10^5 \frac{N}{m^2}$, the temperature 291,8° K. The measurements were carried out with a piston displacement $\delta = 2,55 \times 10^{-3}$ m. From these values we obtain with help of (1.3), (3.8) and (7.12): $\epsilon = 2,3 \times 10^{-2}$, $\sigma = 1,31$, $\omega_0 = 111,3$ sec^{-1}.

At a number of values of ω near ω_0 the pressure at the piston was recorded. With the equipment used for these experiments, it was unfortunately not possible to measure the very small pressure oscillations at the mouth of the pipe and to verify the assumed conditions there. The phase was determined by recording also the path of the piston. A typical pressure recording is shown in figure 7. Since the displacement of the piston was recorded as positive when the piston moved towards the open end, in fact (cf. fig. 1) $- x_{\text{piston}}$ is recorded.

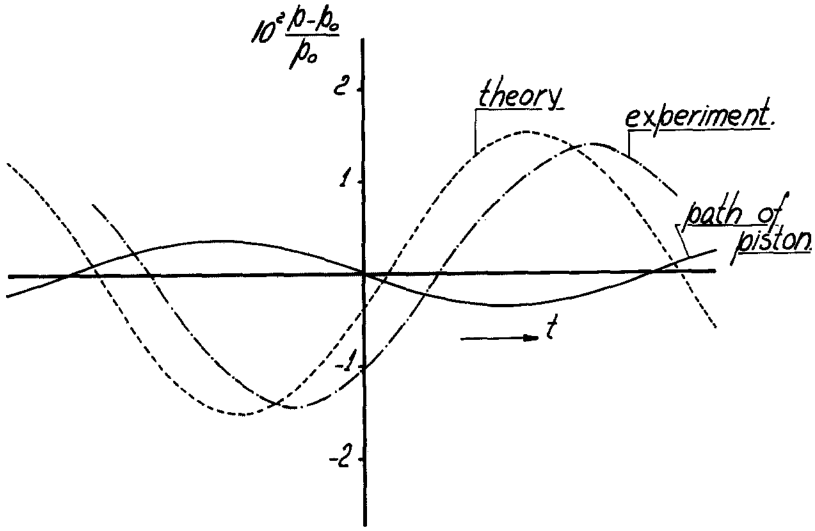


Figure 7: Pressure disturbance, measured at piston for $\frac{\omega_1}{\omega_0} = 1.57$ together with solution obtained with analog computer.

Equation (7.11) was, with the pertinent values for σ and ω_0 solved on an analog computer. For this purpose (7.11) was brought in the form

$$g \frac{dX}{d\tau} + 0.1\, X|X| = \cos \tau ,$$

g being given by

$$g = \left(\frac{\omega_1}{\omega_0} + \frac{\sigma}{2}\right) \left(\frac{\pi}{10}\right)^{\frac{1}{2}}$$

In the example of figure 7 $\frac{\omega_1}{\omega_0} = -1.57$ or (from 6.5) $\frac{\omega_0 - \omega}{\omega_0} = 3,6 \times 10^{-2}$.

The value of g here is -0.52.

The solution for $g = -0.52$, obtained from the solution for $g = 0.52$ by the method described in the foregoing section, is shown also in fig. 7. For the purpose of presentation the scales of the measurements on the scope and of the numerical solution are brought in agreement.

In figure 8 the amplitude of the pressure oscillation at the piston is given as percentage of p_0 both from experiment and theory.

In figure 9 the phase of the pressure oscillation with respect to piston path is given, as obtained from theory and measurements.

Because the pressure oscillation is not a pure sine, the concept of phase lag is not unambiguous. Here it is based on the difference in phase between

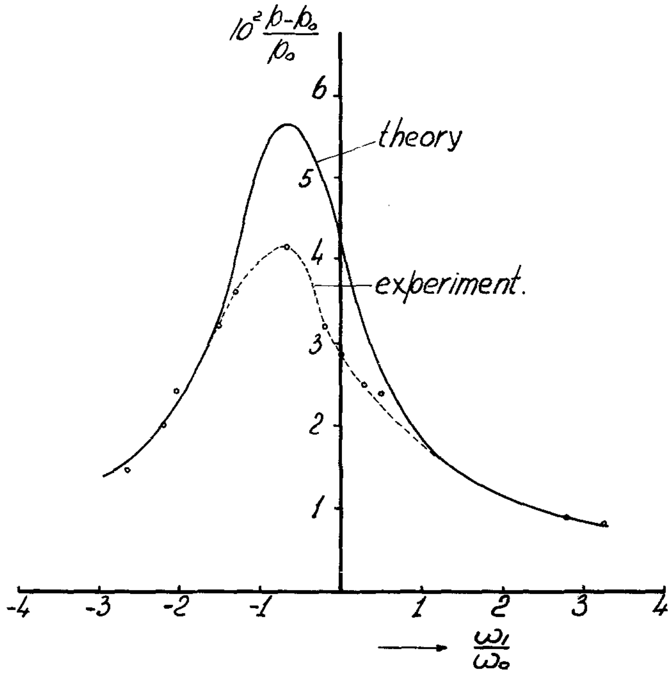


Figure 8: Amplitude of pressure oscillation at the piston as function $\frac{\omega_1}{\omega_0}$. Theory and experiment.

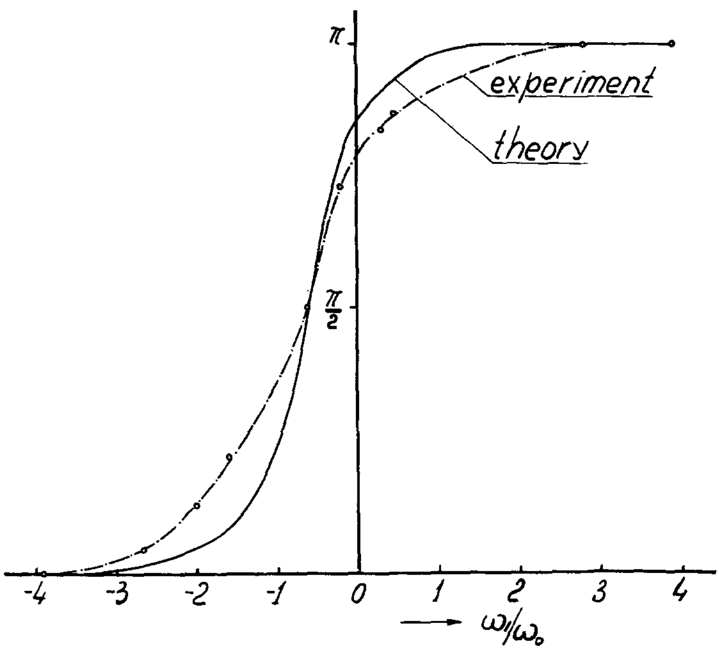


Figure 9: Phase of pressure oscillation with respect to piston path. Theory and experiment.

the zero's. From fig. 9 it follows that in the measurements the phase difference is more near $\pi/2$ than in the theory.

On the basis of (7.8) resonance could be expected at $\frac{\omega_1}{\omega_0} = -0.65$ or (using 6.5) $\frac{\omega}{\omega_0} = 0.985$. The measurements give at resonance (maximum pressure amplitude) $\frac{\omega}{\omega_0} = 0.98$, so that agreement with the resonance frequency given by Rayleigh (used in 7.12) is most satisfactory.

Concerning the amplitude of oscillation, it follows that there is a difference between theory and experiment at and immediately near resonance. This fact, together with the singular behaviour at resonance of $\frac{\partial p}{\partial t}$ for $p - p_0 = 0$, discussed in section 7, indicated that a more accurate account should be given of the conditions prevailing at the mouth of the pipe. While these conditions are not known exactly the present theory is able to predict the frequency and the order of magnitude of the resonance oscillations adequately. For many engineering purposes no more is needed.

Finally something should be said about viscosity effects. These are indirectly taken into account of course in the formulation of the conditions at the mouth of the pipe. The viscosity effects as they are represented in the equation of motion, are neglected in the presented theory since they are of higher order in ϵ. It may be however that the small phase shift due to the action of viscosity plays an inportant part in removing the singularity mentioned above. The pursuing of this point is left for future research.

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